Insights and Puzzles from Branes: $4d$ SUSY Yang-Mills from $6d$ Models

A.Marshakov [*],

Theory Department, P. N. Lebedev Physics Institute, Leninsky prospect 53, Moscow, 117924, Russia

and

ITEP, Moscow, 117259, Russia

M.Martellini [†]

Dipt. di Fisica, Univ. of Milano and I.N.F.N. Sez. di Milano, Via Celoria 16, 20133 Italy

and

A.Morozov [‡]

ITEP, Moscow, 117 259, Russia

Abstract

5-branes of nontrivial topology are associated in the Diaconescu-Hanany-Witten-Witten (DHWW) approach with the Seiberg-Witten (SW) theory of low-energy effective actions. There are two different "pictures", related to the IIA and IIB phases of M-theory. They differ by the choice of $6d$ theory on the 5-brane world volume. In the IIB picture it is just the $6d$ SUSY Yang-Mills, while in the IIA picture it is a theory of SUSY self-dual 2-form. These two pictures appear capable to describe the (non-abelian) Lax operator and (abelian) low-energy effective action respectively. Thus IIB-IIA duality is related to the duality between Hitchin and Whitham integrable structures.

[*]E-mail address: mars@lpi.ac.ru, andrei@heron.itep.ru, marshakov@nbivms.nbi.dk

[†]Landau Network at Centro Volta, Como, Italy

[‡]E-mail address: morozov@vxdesy.desy.de

1 Introduction

According to general principles of string program, various quantum field theory models are identified with the various classical configurations ("vacua") of the string theory, which can be considered as a kind of universal object of the Quantum Field Theory.

Recent advances in this direction appeared due to the introduction of the new class of string vacua: described in terms of the BPS-saturated branes. At present stage the focus of research is on the branes of non-trivial topology. In particular, the system of parallel p-branes appears to be associated with the $p + 1$-dimensional (SUSY) Yang-Mills models [1] and non-perturbative phenomena in the Yang-Mills theory can be reformulated as interactions of branes. This interaction makes the geometry of branes non-flat and in the low-energy limit the nontrivial geometry plays the same role as compactification, thus effectively reducing the naive number of the space-time dimensions. It opens a way for geometrical reinterpretation of the interaction in Yang-Mills theory: the old dream is getting real.

The simplest realization of this idea – the DHWW construction [2, 3, 4] allows one to associate the non-perturbative low-energy p-dimensional SUSY Yang-Mills (SYM) theory with non-trivial vacua of the $p + 1$-dimensional SUSY gauge theory of forms on the brane world volume [1]. This sheds light to the mysteries of the SW theory [5, 6, 7] of the low-energy effective actions (RG-flows) in $N = 2$ SYM theory. In [4] Witten interpreted the SW curve as topologically nontrivial constituent of the brane configuration. In this paper we discuss how the (Toda chain) Lax operator may arise from the DHWW construction for $p = 4$. As anticipated in [8, 9] this can be understood in terms of $p + 1 = 6$-dimensional SUSY Yang-Mills theory on the world volume of the 5-brane. While being adequate for the description of renormalization group (RG) flow from a non-abelian theory in the ultraviolet (UV) limit and of emerging integrable structure, such essentially IIB stringy picture is not enough, however, to obtain a simple description of the prepotential. Instead, this is straightforward in the dual IIA-inspired picture [4] when the SUSY $p + 1 = 6$ dimensional theory on a brane world volume is a theory of self-dual 2-form.

The whole construction is a direct developement of Diaconescu's [2] original reasoning for $p = 1$ (and its analog for $p = 3$), where the Nahm equations [10, 11] arise as non-trivial generalization of the Toda-chain formalism (relevant for $p = 4$). It makes the appearence of a complex spectral curve and prepotential a little more natural – though it is hardly an explanation in intrinsic terms of the SYM theory, and universality of emerging structures is not quite obvious. Even more important, these two ingredients of the SW theory (the curve and the prepotential for the *given* curve) remain linked to two different "pictures" in M-theory. Thus the main dynamical question – of the derivation of the SW ansatz *as a whole* – or [12, 13], of the derivation of abelian Whitham effective low-energy dynamics from the non-abelian Yang-Mills-Hitchin one

---

[1]The role of non-trivial SYM vacua was emphasised in [1]. Technically we seem to overcome the argument of [1] against considering the Dirichlet 5-branes (i.e. against our type-IIB picture) due to nontrivial boundary conditions for the scalar fields, see sect.4 below.

– is not resolved, but reinterpreted as the question about duality between IIA and IIB-type pictures, i.e. is put closer to the main stream of the studies of string dualities.

2   DHWW construction

In the DHWW construction one essentially considers a 5-brane in M-theory [2]. In the first-quantized formalism (still the only one available in most string theory considerations), its dynamics is effectively described by the world-volume ($6d$) theory of either the SYM – in the type IIB picture or the SUSY self-dual 2-form $C = \{C_{MN}\}$, $dC = *dC$ – in the type IIA picture. After compactification on a circle the low-energy theory is $5d$ SYM, and compactification on a Riemann surface (complex curve) Σ leads to a world-volume $4d$ $N = 2$ SYM.

As usual [1], the gauge group $SU(N_c)$ is defined by topology of the brane, and in the low-energy regime it is broken down to the abelian $U(1)^{N_c-1}$, with the scalar (adjoint Higgs) vacuum expectation values identified with (some) moduli of the complex structure on Σ. Most important, in order to allow interpretation in terms of spontaneously broken $SU(N_c)$ gauge symmetry, the choice of the Riemann surface Σ is severely restricted: to hyperelliptic complex curves, being at the same time N_c-fold coverings of a cylinder and associated to the Toda-chain integrable systems. Their appearance in the form of either 2-fold or N_c-fold covering is responsible for two possible descriptions: in terms of $SU(2)$ and $SU(N_c)$ groups – well known both in the brane language [3, 4, 15] and in the approach based on integrable systems [16].

The DHWW construction is essentially as follows [4]: one starts with embedding the 5-brane's world-volume into 11-dimensional target space-time with the co-ordinates $x^0, ..., x^{10}$. We further assume that the target space has topology $R^9 \times T^2 = R^9 \times S^1 \times S^1$. The second S^1 (spanned by x^{10}) will be ignored in what follows[3], while the essential compact co-ordinate will be called x^9. Now, one proceeds with a 5-brane with the world-volume topology $R^5 \times S^1$ and parameterized by $(x^0, x^1, x^2, x^3, x^6, x^9)$, leaves aside four flat dimensions (x^0, x^1, x^2, x^3 – the space-time of the low-energy $4d$ $N = 2$ SYM theory), and ends up with a cylinder $R \times S^1$ embedded into the target space along (x^6, x^9) dimensions. We use the notation $z = x^6 + ix^9$ for the corresponding complex co-ordinate. Next, in order to get a non-trivial gauge group: spontaneously broken $SU(N_c) \rightarrow U(1)^{N_c-1}$, one needs N_c *parallel* copies of the cylinder (see Fig.1). Different cylinders have different positions in "transverse" space $V^{\perp} = (x^4, x^5, x^7, x^8)$. Moreover [2], to get not just a remnant $U(1)^{N_c-1}$, but indeed a spontaneously broken non-abelian theory, these parallel cylinders should come from a *bound state* of N_c branes i.e. should be different parts of the *same* brane [4]. It means that the cylinders should be all glued together (see Fig.2). Actually in the weak coupling limit they are glued at infinity, while

---

[2]Original papers [2] and [3] deal with various degenerations of this construction, when some compactification radia go to zero, thus giving rise to $1, 3, 4$-branes. See also [14] for some important preliminary works and [15] for more examples.

[3] Of course, it is crucially important for accurate embedding of IIA and IIB strings into generic M-theory frame and for the explanation of the *origins* of the two pictures and their interrelation – this is however beyond the scope of present paper.

increasing coupling constant distorts them along their entire length. Since the cylinders are 2-dimensional and parallel, projection of this entire configuration onto $V^\perp$ is also 2-dimensional. Supersymmetry requires it to be just a plane in $V^\perp$: $C = R^2 \in V^\perp$, we will describe it in terms of the complex coordinate $\lambda = x^4 + ix^5$.

Introducing coordinate $w = e^z$ to describe a cylinder, we see that the system of non-interacting branes (Fig.1) is given by z-independent equation

$$P_{N_c}(\lambda) = \prod_{\alpha=1}^{N_c} (\lambda - \lambda_\alpha) = 0, \tag{2.1}$$

while their bound state (Fig.2) is described by the complex curve Σ_{N_c} [4]:

$$\Lambda^{N_c}\left(w + \frac{1}{w}\right) = 2P_{N_c}(\lambda) \qquad \text{or}$$

$$\Lambda^{N_c} \cosh z = P_{N_c}(\lambda) \tag{2.2}$$

In the weak-coupling limit $\Lambda \to 0$ (i.e. $\frac{1}{g^2} \sim \log \Lambda \to \infty$) one comes back to disjoint branes (2.1) [4] [5].

Thus we finally got a 5-brane of topology $R^3 \times \Sigma_{N_c}$ embedded into a subspace $R^5 \times S^1$ (spanned by $x^1, ..., x^6, x^9$) of the full target space. The periodic coordinate is

$$x^9 = \arg P_{N_c}(\lambda) = \operatorname{Im} \log P_{N_c}(\lambda) = \sum_{\alpha=1}^{N_c} \arg(\lambda - \lambda_\alpha) \tag{2.4}$$

3 Lax operator

According to [12], occurence of the complex curves like Σ_{N_c} is a manifestation of hidden integrable structure behind the theory of renormalization group (RG) flows. Namely [12, 20, 21, 22, 23, 24, 25], equation (2.2) describes the spectral curves of the $(0 + 1$-dimensional, N_c-periodic) Toda-chain hierarchy:

$$\det_{N_c \times N_c} (\Lambda\mathcal{L}(z) - \lambda \cdot \mathbf{1}) = 0, \tag{3.1}$$

The $SU(N_c)$ Lax operator

$$\mathcal{L}(z) = \vec{p}\vec{H} + e^{\vec{\alpha}_0\vec{q}}(e^z E_{\vec{\alpha}_0} + e^{-z}E_{-\vec{\alpha}_0}) + \sum_{\text{simple } \vec{\alpha}>0} e^{\vec{\alpha}\vec{q}}(E_{\vec{\alpha}} + E_{-\vec{\alpha}})$$

$$\vec{\alpha}_0 = -\sum_{simple\ \vec{\alpha}>0} \vec{\alpha} \tag{3.2}$$

---

[4]In other words, for $|z| \ll |\log \Lambda|$, λ is almost independent of z and confined to be almost equal to some of λ_α. Only when $|z| \sim |\log \Lambda|$ coordinate λ is allowed to deviate from fixed position and "interpolate" between – different λ_α's.

[5]Eq.(2.2) and Fig.2 decribe a hyperelliptic curve – a double covering of a punctured Riemann sphere,

$$y^2 = \frac{\Lambda^{2N_c}}{4}\left(w - \frac{1}{w}\right)^2 = P_{N_c}^2(\lambda) - \Lambda^{2N_c} \tag{2.3}$$

Such hyperelliptic curves and their period matrices are the main ingredients of the SW ansatz [17, 18, 19] for the 4d $N = 2$ SUSY low-energy effective actions.

where $\vec{H}$ are the diagonal (Cartan) $SU(N_c)$ matrices and $E_{\vec{\alpha}}$ are matrices corresponding to the roots of $SU(N_c)$: $E_{\vec{\alpha}_{ij},mn} = \delta_{mi}\delta_{nj}$. Only the simple roots with $j = i \pm 1$ appear in (3.2). The Hamiltonians of the Toda chain are symmetric polynomials of parameters λ_α in (2.1), e.g.

$$\Lambda^2 h_2 \equiv \Lambda^2 \left(\vec{p}^2 + e^{2\vec{\alpha}_0 \vec{q}} + \sum_{\text{simple } \vec{\alpha} > 0} e^{2\vec{\alpha}\vec{q}} \right) = \sum_{\alpha < \beta} \lambda_\alpha \lambda_\beta \tag{3.3}$$

From the point of view of the DHWW construction the shape of the complex curve Σ_{N_c} should not be just guessed or postulated: it describes the eigenvalues of the scalar field $\Phi(z)$ – the member of the $6d$ supermultiplet, i.e. λ_α's are solutions of the equation

$$\det_{N_c \times N_c} (\Phi(z) - \lambda \cdot \mathbf{1}) = 0, \tag{3.4}$$

which describes mutual positions of the branes, i.e. our cylinders. Thus, comparing (3.1) and (3.4), we conclude that there is a natural identification

$$\Lambda \mathcal{L}(z) \sim \Phi(z) \tag{3.5}$$

This is in fact a general point in the Hitchin approach to integrable systems [26, 27, 28] and this was already used many times in applications of this formalism to investigation of the Seiberg-Witten effective theory [28, 29, 30, 31, 9].

Thus, now we have something to check: the Lax operator (3.2) should naturally arise from the equations of motion for the scalar field $\Phi(z)$. Moreover, in order to preserve supersymmetry, it should satisfy an even more restrictive condition: the linear BPS-like equation.

4 The IIB type picture

In order to explain how it happens, let us analyze the DHWW construction in the type IIB picture, when the theory on the 5-brane world volume is $6d$ SYM. The mutual position of the cylinders on Fig.1 is described by the coordinates in orthogonal space $V^\perp$ (spanned by x^4, x^5, x^7, x^8), i.e. by four scalar fields $\Phi^{(4)}, \Phi^{(5)}, \Phi^{(7)}, \Phi^{(8)}$ – the members of the $6d$ SYM gauge multiplet. As usual, they are taking values in the adjoint representation of the gauge group $SU(N_c)$, where N_c is the number of cylinders, i.e. co-ordinates $x^4, \ldots, x^8$ are substituted by non-obligatory commuting matrices $\Phi^{(4)}, \ldots, \Phi^{(8)}$. The members of gauge multiplet are associated with the open strings streched between the cylinders, the corresponding $10d$ vector field $A_S = \{A_M, A_\mu\}$ in the bulk naturally decomposes into the components with $M = 0, 1, 2, 3, 6, 9$ – considered as $6d$ vector from the point of view of the effective theory on the brane – and with $\mu = 4, 5, 7, 8$ – associated with four above-mentioned scalars. The nonabelian interaction arises due to the processes like in Fig. 3.

In a vacuum state the scalar fields satisfy the BPS-like condition

$$D_M \Phi \equiv \partial_M \Phi + [A_M, \Phi] = 0, \quad F_{MN} = 0 \tag{4.1}$$

This equation is so simple at least when only one of the fields $\Phi^{(4)}, \ldots, \Phi^{(8)}$ is nonvanishing. This is essentially the case for the configuration of Fig.2, arising from Fig.1 when the brane interaction is switched on: Fig.2 implies that some scalar field, say $\Phi \equiv \Phi^{(4)} + i\Phi^{(5)}$, develops a nonvanishing z-dependent vacuum expectation value – this is exactly the statement that the cylinders are distorted and glued together. In order to explain/derive Fig.2, it is necessary to demonstrate that eq. (4.1) has a *non-trivial* solution $\Phi(z) \neq const.$ The reason for this is that non trivial boundary conditions are imposed on Φ at $z \to \pm\infty$.

In order to understand how they should be adequately described, let us consider first the UV-finite version of the SW theory and then take the double-scaling limit back to the asymptotically free situation.

The way to do this is well known and examined in detail in [6, 28, 31]. One should add to the $4d$ $N = 2$ SUSY pure gauge theory an extra matter hypermultiplet in the adjoint representation with the mass m. When $m = 0$, one gets a theory with $N = 4$ SUSY which is UV-finite with the UV coupling constant $\tau = \frac{i}{g_{UV}^2} + \frac{\theta}{2\pi}$. When $m \neq 0$ it remains UV-finite, but acquires a nontrivial RG-flow. The original pure $N = 2$ SYM theory is restored in the double-scaling limit when $\tau \to i\infty$, $m \to \infty$, so that $m^{N_c}e^{2\pi i\tau} \equiv \Lambda^{N_c}$ remains finite. Within the framework of the SW theory this corresponds to a spectral curve – a cover of a torus (with complex modulus τ) which in the limit $\tau \to i\infty$ degenerates into a cylinder. Associated integrable system is the elliptic Calogero-Moser model with the coupling constant m [32, 29, 30, 31] which in the double-scaling limit [33] turns into a Toda chain.

From the point of view of the brane picture at Fig.1 it means that one should first substitute the cylinders by tori with the same modulus τ. The isolated tori would correspond to the vanishing parameter m, while non-vanishing m means that the scalar field Φ acquires nontrivial boundary consitions, or is a section of a nontrivial (holomorphic) bundle. In other words, when one takes cylinders from Fig.1 and glues the ends to make a torus – the fields jump, and on the torus the equation (4.1) acquires a non-zero r.h.s., which survives in the double-scaling limit.

More technically, on a torus one cannot fix the gauge $\bar{A} \equiv A_6 + iA_9 = 0$, by gauge transformation $\bar{A}$ can be at best brought to diagonal form $\bar{A} = diag(a_1, ..., a_{N_c})$. Then the corresponding component of equation (4.1) becomes[6]

$$\bar{\partial}\Phi^{ij} + (a_i - a_j)\Phi^{ij} = m(1 - \delta^{ij})\delta(z - z_0) \tag{4.2}$$

so that

$$\Phi^{ij}(z) = p_i\delta^{ij} + m(1 - \delta^{ij})e^{(a_i - a_j)(z - \bar{z})}\frac{\theta(z - z_0 + \frac{a_i - a_j}{\pi \mathrm{Im}\tau})}{\theta(z - z_0)} \tag{4.3}$$

To compare with the conventional Lax operator of the elliptic Calogero-Moser model [34], one should make a gauge transformation

$$\Phi^{ij}(z) \to (U^{-1}\Phi U)^{ij}(z) = p_i\delta^{ij} + m(1 - \delta^{ij})e^{(a_i - a_j)z}\frac{\theta(z - z_0 + \frac{a_i - a_j}{\pi \mathrm{Im}\tau})}{\theta(z - z_0)} \tag{4.4}$$

---

[6]There are different ways to interpret the δ-function in the r.h.s. of (4.2): one can say, for example [9], that $z = z_0$ is the point where "vertical brane" of original presentation of [2, 3] intersect the "horisontal branes" – our tori. In Fig.2 the point z_0 is at infinity, i.e. exactly where nontrivial boundary conditions are imposed in eq.(4.1).

with $U^{ij} = e^{(a_i - a_j)\bar{z}}$, then the explicit dependence on $\bar{z}$ is eliminated but $\Phi(z)$ becomes a multivalued function or a section of a nontrivial bundle over torus. The Lax operator (3.2) is obtained from (4.4) in the double-scaling limit [33].

Of course, there is a way to describe relevant nontrivial boundary conditions directly in terms of Fig.2 (without additional compactification-/-decompactification of the x^6 dimension), but the above presentation reveals better the origins of what happens.

Thus, in the type IIB picture we derived the *shape* of the curve (2.2), (3.1), (3.4) "from the first principles". The next step would be to derive the effective action of emerging low-energy $4d$ theory. However, here one runs into problems.

Since the world-volume action is not quadratic, it is necessary to take non-trivial average over the fields which become massive due to the Higgs mechanism, moreover this average includes non-perturbative corrections. This is more or less the same as the original problem in the SW theory, without any obvious simplifications. As explained in [12, 23], from the point of view of integrable hierarchies the derivation of the low-energy effective action is the aim of the so-called Bogolyubov-Whitham averaging method, which is still far from being throughly developed. Remarkably, despite such problems, the net *result* of this procedure can be easily described in terms of period integrals on spectral surfaces, i.e. in the framework of the prepotential theory (or that of the quasiclassical τ-functions) [35, 36, 37], which is in a sense "dual" to the Hitchin theory [23, 38].

This is exactly in parallel to what happens in SW theory: while there is no clear way to *derive* effective action directly, the ansatz can be easily suggested for what it actually is. In other words, despite the brane vacuum configuration is derived exactly in the type IIB picture, this picture is not sufficient itself for the derivation of the effective action (at least it is not straightforward). However, according to [4], this problem can be solved in the "dual" type IIA picture.

5   IIA type picture

In this picture instead of the $6d$ SYM one considers a $6d$ SUSY theory of self-dual 2-form $C = \{C_{MN}\}$, $dC = *dC$ on the world volume of a 5-brane. It means, first, that instead of attaching open strings to the 5-brane, as in Fig.3, one has to consider now "open" membranes, see Fig.4.

The important difference with the type IIB picture of the previous section is that in the relevant approximation the theory of 2-forms is essentially abelian. Even if there are matrices C_{MN}^{ij} in the adjoint representation of $SU(N_c)$ associated with the vertical cylinders (membranes) attached between i-th and j-th horisontal cylinders, no nonabelian interacting theory can arise since such interaction is inconsistent with the gauge invariance. Only non-linear interaction of the non-minimal type can appear – like $\text{Tr}(dC)^4$, expressed through the tension of C. Such terms, however, contain higher derivatives (powers of momentum) and they

seem irrelevant in the low-energy effective actions.

This "abelian" nature of the 2-form theory makes the description of the Lax operator (vacuum expectation value of the scalar members of the supermultiplet which describe the transverse fluctuations of the 5-brane), and thus the derivation of the shape of the curve Σ_{N_c} in the type IIA picture, much less straightforward. Instead, exactly due to the fact that the action on (flat) world-volume is essentially quadratic

$$\int_{d^6x} |dC|^2 + \text{SUSY terms}, \tag{5.1}$$

in this picture there are no corrections to the form of the effective $4d$ action – once Σ_{N_c} is given. It is enough to consider the dimensional reduction of (5.1) from 6 to 4 dimensions [4].

Such reduction implies that the 2-form C is decomposed as

$$C_{\mu z} = \sum_{i=1}^{N_c-1} \left(A_\mu^i(x)d\omega_i(z) + \bar{A}_\mu^i(x)d\bar{\omega}_i(\bar{z}) \right) \tag{5.2}$$

where $d\omega_i$ are canonical holomorphic 1-differentials on Σ_{N_c} [7], $d\bar{\omega}_i$ – their complex conjugate, and A_μ^i, $\bar{A}_\mu^i$ depend only on the four $4d$ co-ordinates $x = \{x^0, x^1, x^2, x^3\}$.

If the metric on Σ_{N_c} is chosen so that $*d\omega_i = -d\omega_i$, $*d\bar{\omega}_i = +d\bar{\omega}_i$, the self-duality of C implies that the 1-forms A and $\bar{A}$ in (5.2) correspond to the anti-selfdual and selfdual components of the $4d$ gauge field with the curvature (tension) $G = \{G_{\mu\nu}\}$:

$$dA^i = G^i - *G^i$$

$$\tag{5.3}$$

$$d\bar{A}^i = G^i + *G^i$$

It remains to subsitute this into (5.1) and use the relations

$$\int_{\Sigma_{N_c}} d\omega_i \wedge d\bar{\omega}_j = 2i\text{Im}T_{ij}$$

$$\tag{5.4}$$

$$\int_{\Sigma_{N_c}} d\omega_i \wedge d\omega_j = 0$$

where T_{ij} is the period matrix of Σ_{N_c} (and depends on the v.e.v.'s of the transverse scalar fields once the shape of the curve Σ_{N_c} – its embedding into the (x^4, x^5, x^6, x^9)-space is fixed. The result for the $4d$ effective action is

$$\int_{d^4x} \text{Im}T_{ij}G_{\mu\nu}^i G_{\mu\nu}^j + \text{SUSY terms} \tag{5.5}$$

---

[7]Actually, before the double scaling limit described in the previous section, the curve Σ_{N_c} is compact of genus N_c (when the x^6 direction is also compactified along with x^9) and the curve possesses N_c holomorphic differentials. However, one of them develops a simple pole when $z \to \infty$ and thus is ignored in (5.1). Also in our simplified description we ignore other components of the 2-form: $C_{z\bar{z}}$ and $C_{\mu\nu}$ which are related to each other by the selfduality condition

$$\partial_\lambda C_{z\bar{z}} = \frac{1}{\sqrt{g}}\epsilon_{\lambda\mu\nu\rho}\partial_\rho C_{\mu\nu}$$

and correspond from the $4d$ point of view to a (real) scalar. The whole picture thus would contain *three complex* scalar fields, two of which become massive in the configuration we consider.

8

This is essentially the SW answer of refs. [5, 6], only the part with the topological θ-term is ignored. It can be restored by more careful treatment of the self-dual 2-forms: the action (5.1) is obviously too naive. There are various approaches developed for this purpose, see for example refs. [39, 40, 41, 42, 43] etc.

Though in principle dictated by supersymmetry, the analog of expansion (5.2) is not so trivial for the scalar fields (the superpartners of the antisymmetric form in $6d$ and the vector bosons in $4d$). As usual in the Green-Schwarz formalism on topologically non-trivial manifolds [44], like Σ_{N_c}, the "embedding matrix" $\Pi_z = \partial_z \Phi$ is actually substituted by 1-form on Σ, with holomorphic zero modes,

$$\Pi_z^{(0)} = \sum_{i=1}^{N_c-1} \Pi^i(x) d\omega_i(z) \tag{5.6}$$

This is important for explanation why $N_c - 1$ different scalars emerge from a single Φ, and why the period matrix of Σ appears in the scalar Lagrangian. Especially transparent should be (reformulation of) the formalism of ref.[42], where appropriate auxiliary field is actually a 1-form on Σ_{N_c}, which after gauge fixing becomes

$$v_z^{(0)} = \sum_{i=1}^{N_c-1} v^i(x) d\omega_i(z) \tag{5.7}$$

6 Conclusion

We argued that the recent advance of ref.[4] (which reformulated the SW anzatz in the language of branes and therefore inspired an anzatz for what the interaction of branes does with the naive DHW construction) still does no resolve the basic problem of all previous considerations: two basic different ingredients of the SW theory (the spectral curve and the prepotential) are well justified in two dual pictures. However, it brings the issue even closer to the main mysteries of string dualities.

In particular it helps to approach the (still) anticipated discovery of integrable structures behind the string dynamics. In this framework one expects extrapolation of the known results for $2d$, $3d$ and $4d$ models to higher dimensions. As to the *origin* of the integrability in the theory of renormalization group flows, it is a subject of a different investigation. A possible direction has been suggested in ref.[12], and emerging relation between the IIB-IIA and the ("nonabelian") Hitchin - ("abelian") Whitham dualities can provide new insights on this way.

However, even in the restricted framework – of generalization of the SW theory to higher dimensions, strings and M-theory – a lot remains obscure. One of the interesting things to do is to find the brane analog/interpretation of the mysterious WDVV-like equations [45], which are peculiar for the majority of SW effective theories in four and five dimensions, are related to multiplication "algebra" of 1-forms and constitute a non-trivial deformation of the WDVV equations for quantum cohomologies [46].

7   Acknowledgements

References

[1] E.Witten, Nucl.Phys. B460 (1996) 335, hep-th/9510135.

[2] D.-E.Diaconescu, hep-th/9608163.

[3] A.Hanany and E.Witten, hep-th/9611230.

[4] E.Witten, hep-th/9703166.

[5] N.Seiberg and E.Witten, Nucl.Phys. B426 (1994) 19, hep-th/9407087.

[6] N.Seiberg and E.Witten, Nucl.Phys. B431 (1994) 484, hep-th/9408099.

[7] N.Seiberg and E.Witten, hep-th/9607163.

[8] E.Martinec and N.Warner, hep-th/9511052.

[9] A.Gorsky, hep-th/9612238.

[10] W.Nahm, Phys.Lett. 90B (1980) 413.

[11] E.Corrigan and P.Goddard, Ann. of Phys. 154 (1984) 253.

[12] A.Gorsky, et al., Phys.Lett. B355 (1995) 466, hep-th/9505035.

[13] See also refs.[23, 38, 9] and papers
A.Faraggi and M.Matone, Phys.Rev.Lett. 78 (1997) 163, hep-th/9606063;// R.Carroll, hep-th/9702138; hep-th/9705229.

[14] P.Townsend, Phys.Lett. B373 (1996) 68-75, hep-th/9512062;
T.Banks, M.Douglas and N.Seiberg, Phys.Lett. B387 (1996) 278, hep-th/9605199;
M.Douglas and M.Li, hep-th/9604041;

M.Douglas, hep-th/9604198;

G.Chalmers and A.Hanany, Nucl.Phys. B489 (1997) 223-244, hep-th/9608105.

[15] J.de Boer, K.Hori, H.Ooguri and Y.Oz, hep-th/9611063;

O.Aharony, J.Sonnenschein, S.Theisen and S.Yankielowicz, hep-th/9611222;

S.Elitzur, A.Giveon and D.Kutasov, hep-th/9702014;

J.de Boer, K.Hori, Y.Oz and Zh.Yin, hep-th/9702154;

H.Ooguri and C.Vafa, hep-th/9702180;

J.Barbon, hep-th/9703051;

A.Brandhuber, J.Sonnenschein, S.Theisen and S.Yankielowicz, hep-th/9704044;

S.Elitzur, A.Giveon, D.Kutasov, E.Rabinovici and A.Schwimmer, hep-th/9704104;

O.Aharony and A.Hanany, hep-th/9704170;

B.Kohl, hep-th/9705031;

Ch.Ahn and R.Tatar, hep-th/9705106;

K.Landsteiner, E.Lopez and D.Lowe, hep-th/9705199;

A.Brandhuber, J.Sonnenschein, S.Theisen and S.Yankielowicz, hep-th/9705232.

[16] A.Gorsky et al., Phys.Lett. B380 (1996) 75, hep-th/9603140; hep-th/9604078.

[17] A.Klemm, W.Lerche, S.Theisen and S.Yankielowicz, Phys.Lett. B344 (1995) 169, hep-th/9411048.

[18] P.Argyres and A.Farragi, Phys.Rev.Lett. 74 (1995) 3931, hep-th/9411057.

[19] A.Hanany and Y.Oz, Nucl.Phys. B452 (1995) 283, hep-th/9505074.

[20] E.Martinec and N.Warner, Nucl.Phys. B459 (1996) 97, hep-th/9509161.

[21] T.Nakatsu and K.Takasaki, Mod.Phys.Lett. A11 (1996) 157-168, hep-th/9509162.

[22] H.Itoyama et al., hep-th/9601168.

[23] A.Marshakov, Mod.Phys.Lett. A11 (1996) 1169-1184, hep-th/9602005; Int.J.Mod.Phys. A12 (1997) 1607, hep-th/9610242; hep-th/9702083.

[24] R.Donagi, alg-geom/9705010.

[25] A.Klemm, hep-th/9705131.

[26] N.Hitchin, Duke.Math.J. 54 (1987) 91.

[27] A.Gorsky and N.Nekrasov, Nucl.Phys. B436 (1995) 582, hep-th/9401017

[28] R.Donagi and E.Witten, Nucl.Phys. B460 (1996) 299, hep-th/9510101.

[29] E.Martinec, Phys.Lett. B367 (1996) 91, hep-th/9510204.

[30] A.Gorsky and A.Marshakov, Phys.Lett. B375 (1996) 127, hep-th/9510224.

[31] H.Itoyama et al., Nucl.Phys. B477 (1996) 855, hep-th/9511126.

[32] A.Gorsky and N.Nekrasov, hep-th/9401021.

[33] V.Inozemtsev, Comm.Math.Phys. 121 (1989) 629.

[34] I.Krichever, Func.Anal. and Apps. 14 (1980) 282.

[35] I.Krichever, hep-th/9205110; Comm.Math.Phys. 143 (1992) 415.

[36] B.Dubrovin, Nucl.Phys. B379 (1992) 627; Comm.Math.Phys. 145 (1992) 195.

[37] H.Itoyama et al., Nucl.Phys. B498 (1997) 529, hep-th/9512161.

[38] A.Marshakov, hep-th/9607159; hep-th/9608161.

[39] E.Verlinde, Nucl.Phys. B455 (1995) 211, hep-th/9506011.

[40] E.Witten, hep-th/9610234.

[41] J.Schwarz, hep-th/9701008; hep-th/9705092.

[42] P.Pasti, D.Sorokin and M.Tonin, hep-th/9701037;
I.Bandos, K.Lechner, A.Nurmagambetov, P.Pasti, D.Sorokin and M.Tonin, hep-th/9701148; hep-th/9703127.

[43] P.Howe, E.Sezgin and P.West, hep-th/9702008; hep-th/9702111.

[44] R.Kallosh et al., Int.J.Mod.Phys. A3 (1988) 1943; Phys.Lett. B207 (1988) 164.

[45] A.Marshakov, A.Mironov and A.Morozov, Phys.Lett. B389 (1996) 43-52, hep-th/9607109; hep-th/9701014; hep-th/9701123.

[46] E.Witten, Surveys Diff.Geom. 1, (1991) 243;
R.Dijkgraaf, E.Verlinde and H.Verlinde, Nucl.Phys. B352 (1991) 59;
B.Dubrovin, hep-th/9407018;
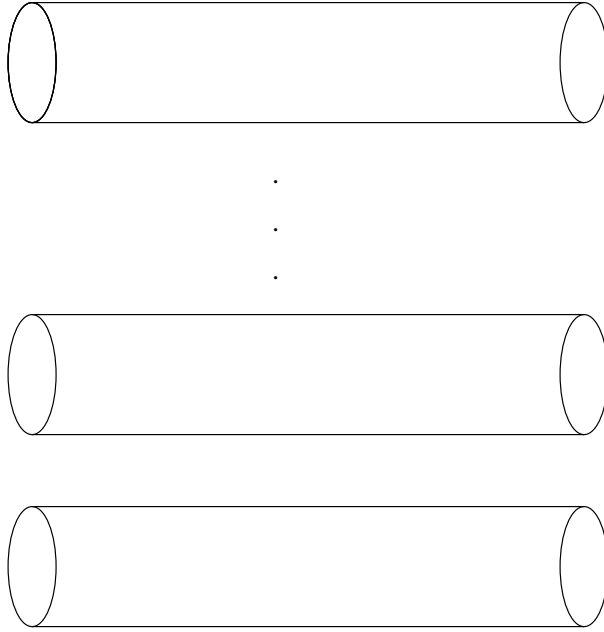Yu.Manin, Lectures at Max Planck Institute, Bonn 1996.

Fig.1

N_c parallel cylinders. The horisontal co-ordinate is x^6, while the vertical axis corresponds to the space $V^\perp$, actually parameterized by λ.
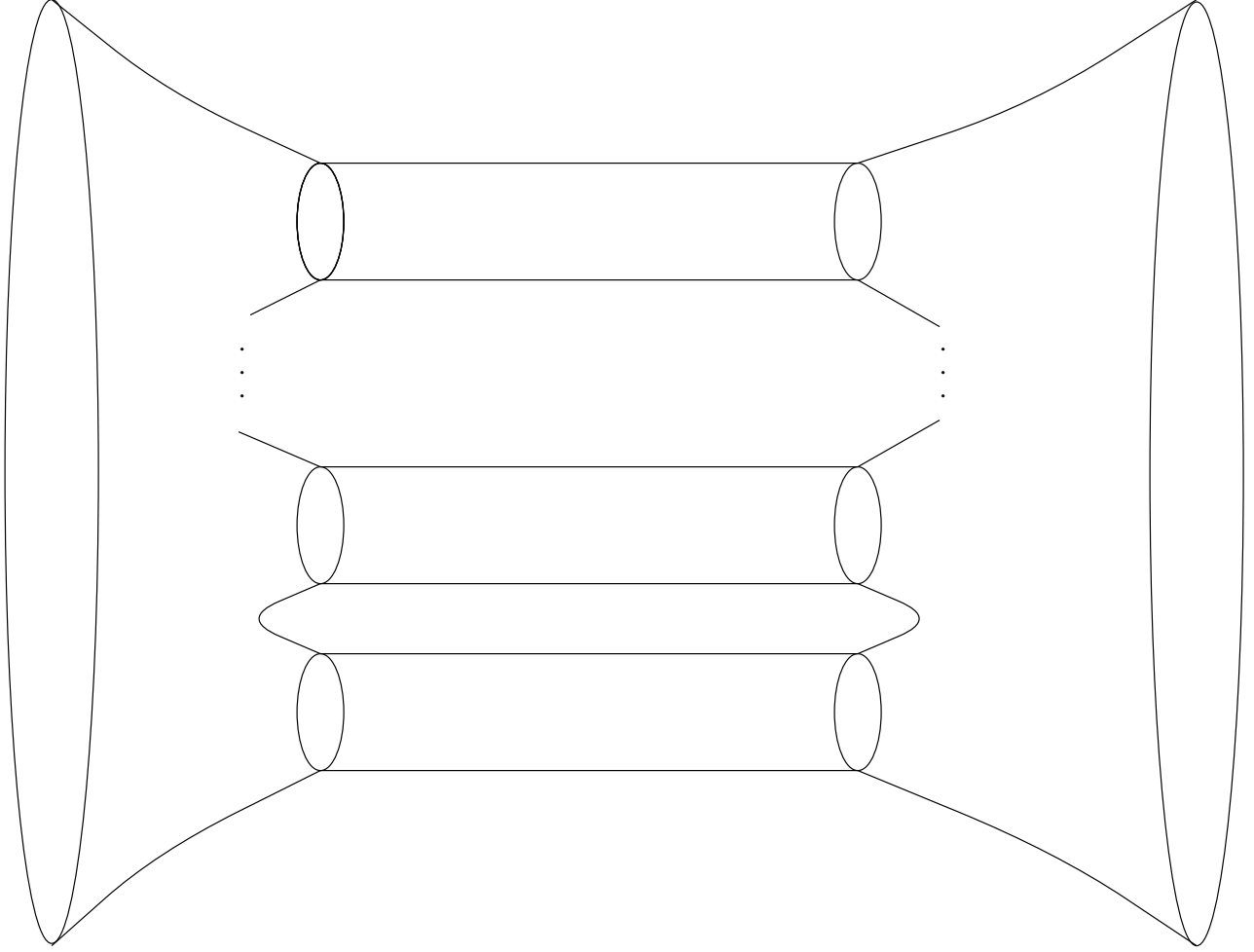
Fig.2

The brane configuration, represented as a result of gluing N_c cylinders together. Actually, a real-λ section of the complex curve (2.2) is shown. The horizontal coordinate is z, the vertical one – λ. If projected on the vertical plane, the curve looks like a double-covering of a punctured Riemann sphere – the hyperelliptic surface. If projected on the horizontal cylinder, it is its N_c-fold covering.
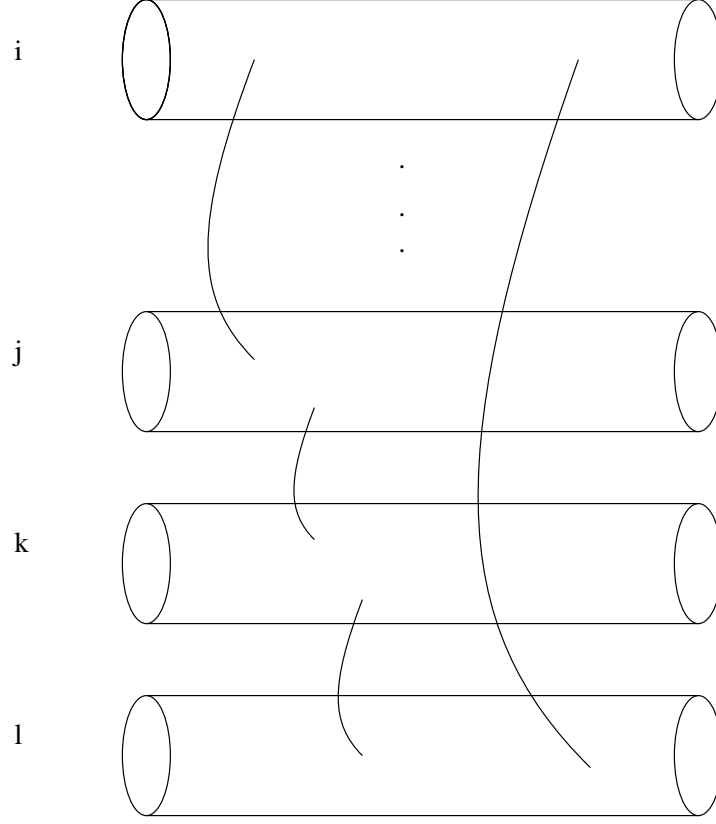
Fig.3

Open strings, stretched between parts of the 5-Dbrane. The term in the $6d$ first-quantized action, associated with this picture is $\delta^{M_1 \cdots M_4} \int d^6 x A_{M_1}^{ij} A_{M_2}^{jk} A_{M_3}^{kl} A_{M_4}^{li} = \text{Tr} \left([A_M, A_N]^2 \right)$.
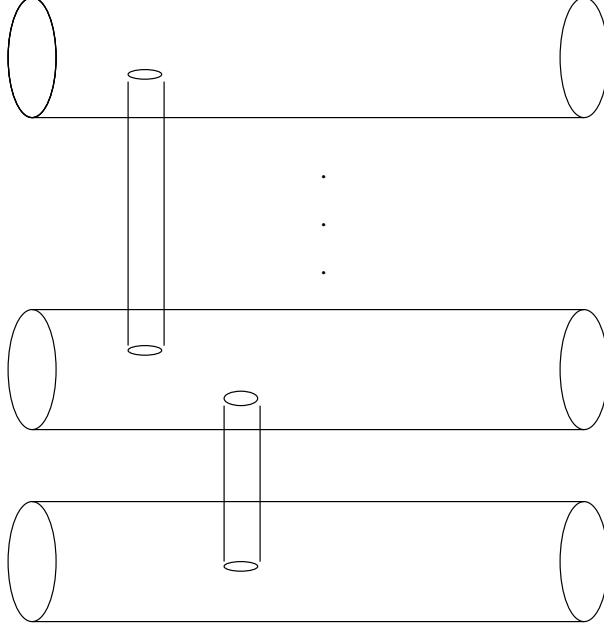
Fig.4

The horisontal cylinders in Fig.4 are parts of the 5-brane, as in Fig.1, with 4 flat dimensions (including the "time"-one x^0) not shown on the picture. The vertical cylinders on Fig.4 are membranes (2-branes) at a given time, streched between the components of the 5-brane. When the width of the vertical cylinders goes to zero, these membranes turn into the open strings. Moreover, this limit is not just a result of shrinking some compact dimension – it is rather a double-scaling limit, when the width of the horisontal cylinders remains finite (and is determined in terms of Λ in (2.2)).